f_1 = function defined by Equation (25)
f_2 = function defined by Equation (26)
g = gravitational acceleration
$\overline{h}$ = basic film thickness
i = $\sqrt{-1}$
N_{Oh} = Ohnesorge number = $\mu/(\rho\sigma\overline{h})^{1/2}$
N_{Re} = Reynolds number = $\overline{u}_s\overline{h}/\nu$
N_ζ = surface tension group = $(\sigma/\rho)(2/g\nu^4)^{1/3}$
R = radius of cylinder
t = time nondimensionalized with respect to $\overline{h}^2/\nu$
$\overline{u}$ = basic flow velocity nondimensionalized with respect to $\overline{u}_s$
$\overline{u}_s$ = surface velocity of basic flow
x = streamwise coordinate nondimensionalized with respect to $\overline{h}$
y = cross-stream coordinate nondimensionalized with respect to $\overline{h}$

Greek Letters

α = wave number = $2\pi\overline{h}/\lambda$
α_m = most highly amplified wave number
α_n = neutrally stable wave number
η = function defined by Equation (7)
λ = wavelength
Λ = curvature group = $R/\overline{h}$
μ = shear viscosity
ν = kinematic viscosity
ρ = density
σ = surface tension
ϕ = amplitude of stream function nondimensionalized with respect to $\sigma\overline{h}^2/\nu$
ϕ_k = k^{th}-order solution for amplitude of stream function

Superscripts

= order of differentiation with respect to y

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The Equivalence of the Spatial and Temporal Formulations for the Linear Stability of Falling Film Flow

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The tendency of falling film flow to be unstable at very low Reynolds numbers leads to the inevitable appearance of ripples. Engineers have been interested in the stability of this flow because of the profound effect these ripples can have on heat and mass transfer rates. The initial linear stability analyses of this flow were developed for temporally growing disturbances whose stream function is given by $\psi = \phi(y)\cdot\exp[i(\alpha_r x - \omega t)]$ and $\omega = \omega_r + i\omega_i$. Recently, this problem has been solved for spatially growing disturbances whose stream function is given by $\psi = \phi(y)\cdot\exp[i(\alpha x - \omega_r t)]$, where $\alpha = \alpha_r + i\alpha_i$.

The question arises as to whether the predictions of the spatial and temporal formulations are equivalent for all observable disturbances. Prior attempts to resolve this question have been inconclusive, as has been discussed by Lin (1975*a*) and Krantz (1975*a*, 1975*b*), either because they employ nonsystematic methods of solution whose range of validity cannot be assessed, or because they are restricted to weakly amplified disturbances for which a transformation of Gaster (1965) assures the equivalence of the two formulations. This transformation permits one to convert the temporal amplification factor into a corresponding spatial amplification factor. Gaster's development is in essence an expansion of the characteristic equation for the linear stability problem about the neutral stability condition, at which of course the predictions of the temporal and spatial formulations are identically equivalent. Thus, this transformation of Gaster is restricted to weakly amplified waves.

The purpose of this note is to compare the predictions of the spatial and temporal formulations of this linear stability problem solved via a systematic method of solution which is not necessarily restricted to weakly amplified long waves.

THEORETICAL DEVELOPMENT

The spatial formulation of the dimensionless Orr-Sommerfeld equation and associated boundary and kinematic conditions appropriate to falling film flow is given by

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha N_{Re}\left[\left(U - \frac{\omega_r}{\alpha}\right)(\phi'' - \alpha^2\phi) - U''\phi\right] \quad (1)$$

$$\phi = 0, \quad \text{at} \quad y = 1 \quad (2)$$

$$\phi' = 0, \quad \text{at} \quad y = 1 \quad (3)$$

$$\phi'' + \left[\alpha^2 - \frac{3}{\left(\frac{\omega_r}{\alpha} - \frac{3}{2}\right)}\right]\phi = 0, \quad \text{at} \quad y = 0 \quad (4)$$

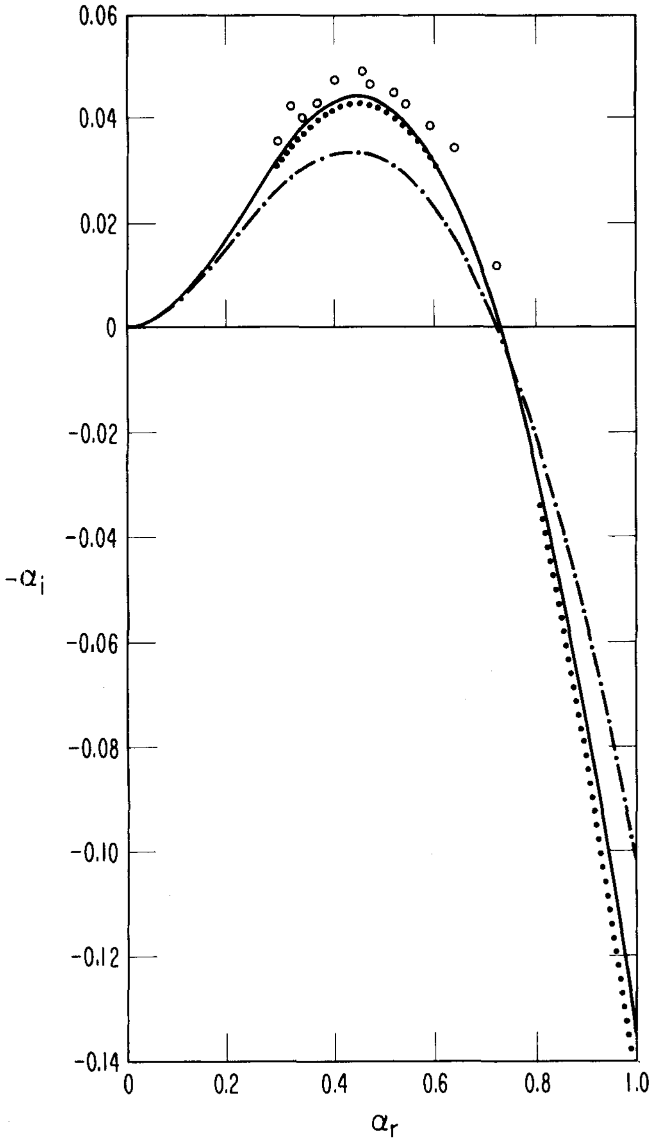

Fig. 1. Spatial amplification factor vs. wave number for $N_{Re} = 1.29$, $N_\zeta = 2.89$, and $\theta = 90$ deg. ○ data of Krantz and Goren (1971); ——— spatial formulation; ...... temporal formulation transformed via Equation (9); —.—.—. temporal formulation transformed via Equation (10).

$$\left[\alpha(3\cot\theta + \alpha^2 N_\zeta N_{Re}^{-2/3})/\left(\frac{\omega_r}{\alpha}-\frac{3}{2}\right)\right]\phi$$

$$-\left[\alpha N_{Re}\left(\frac{\omega_r}{\alpha}-\frac{3}{2}\right)+3\alpha^2 i\right]\phi' + i\phi''' = 0 \quad \text{at} \quad y=0$$

$$\phi = \left(\frac{\omega_r}{\alpha}-\frac{3}{2}\right), \quad \text{at} \quad y=0 \qquad (6)$$

Equations (2) through (6) are the no-flow, no-slip, tangential stress, normal stress, and kinematic surface condition, respectively.

Solution to Equations (1) through (6) describing spatially growing waves can be developed analogously to that of Benjamin (1957) for temporally growing waves. Following Benjamin, we seek a solution of the form

$$\phi = \sum_{N=0}^{\infty} A_N y^N \qquad (7)$$

The solution to Equation (1) given by Equation (7) will be convergent for all physically significant conditions.

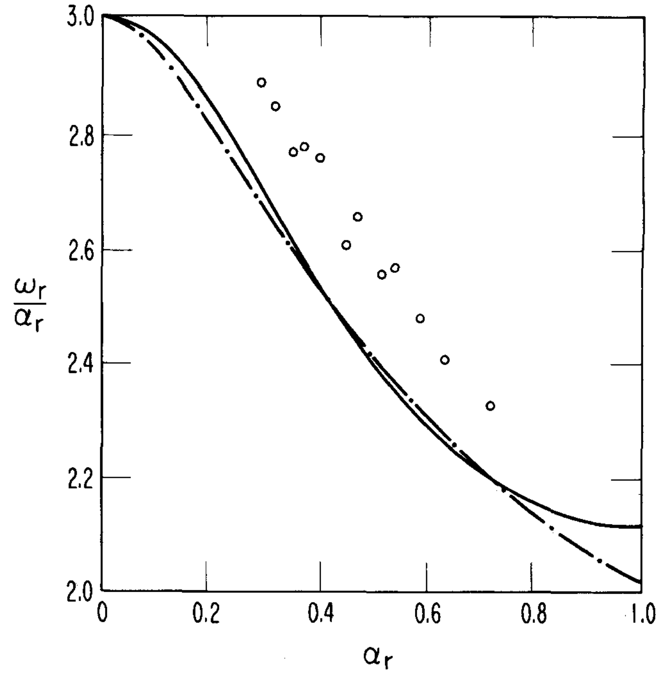

Fig. 2. Phase velocity vs. wave number for $N_{Re} = 1.29$, $N_\zeta = 2.89$, and $\theta = 90$ deg.: ○ data of Krantz and Goren (1971); ——— spatial formulation; —.—.—.—. temporal formulation.

When Equation (7) is substituted into Equation (1), a recurrence formula is generated relating the coefficients A_N for $N > 3$ to the coefficients A_0, A_1, A_2, and A_3. Equations (2) through (6) constitute five equations in these four undetermined constants. Hence a relationship is generated between the parameters of this eigenvalue problem from which the complex eigenvalues α can be determined:

$$\mathrm{i}\left[\frac{2}{3}\alpha^3 N_\zeta N_{Re}^{-2/3} + 2\alpha\cot\theta\right] + 2(\omega_r/\alpha) - 6$$

$$+ i\frac{3}{2}\alpha N_{Re}\left[-\frac{8}{15}(\omega_r/\alpha)^2 + \frac{136}{105}(\omega_r/\alpha) - \frac{24}{35}\right]$$

$$+ \alpha^2[3.6(\omega_r/\alpha) - 4.8] + \frac{9}{4}\alpha^2 N_{Re}^2[0.0016927(\omega_r/\alpha)^3$$

$$- 0.009594(\omega_r/\alpha)^2 + 0.0169753(\omega_r/\alpha) - 0.0102907]$$

$$+ i\frac{3}{2}\alpha^3 N_{Re}[-0.072381(\omega_r/\alpha)^2 + 0.2020600(\omega_r/\alpha)$$

$$- 0.1397615] + \alpha^4[-0.0914284(\omega_r/\alpha) + 0.0742865]$$

$$- \mathrm{i}\frac{27}{8}\alpha^3 N_{Re}^3[-0.0000502(\omega_r/\alpha)^4 + 0.0003382(\omega_r/\alpha)^3$$

$$- 0.0008003(\omega_r/\alpha)^3 + 0.008276(\omega_r/\alpha) - 0.0003175]$$

$$- \frac{9}{4}\alpha^4 N_{Re}^2[0.0021918(\omega_r/\alpha)^3 - 0.0106906(\omega_r/\alpha)^2$$

$$+ 0.0171370(\omega_r/\alpha) - 0.0092402]$$

$$+ \mathrm{i}\frac{3}{2}\alpha^5 N_{Re}[-0.0225191(\omega_r/\alpha)^2 + 0.064464(\omega_r/\alpha)$$

$$- 0.0468239] + \alpha^6[0.0364873(\omega_r/\alpha) - 0.0504137] = 0 \qquad (8)$$

Equation (8) is an approximate solution obtained by retaining sixteen terms in Equation (7). This is equivalent to an asymptotic solution in α retaining terms of sixth order in α under the ordering arguments $N_{Re} = 0(\alpha)$ and $(\omega_r/\alpha) = 0(1)$. Assuming the ordering arguments $N_\zeta = 0(1/\alpha^2)$, $N_{Re} = 0(1)$, $(\omega_r/\alpha) = 0(1)$, and $|\alpha_i| << \alpha_r$, and retaining only first-order terms in α, reduces the above to the asymptotic solution of Agrawal and Lin (1975).

For specified values of N_ζ, N_{Re}, θ, and ω_r, the seven roots of Equation (8) for the complex eigenvalue α can be obtained numerically. The root for which $\alpha_r \geqq 0$, having the largest positive spatial amplification factor $-\alpha_i$, is assumed to be the physically observable root. The interested reader desiring more details on this solution is referred to the thesis of Shuler (1974).

DISCUSSION

The temporal analogue of Equation (8), given by Equation (4.11) in Benjamin (1957), can be obtained from Equation (8) by replacing α with α_r and ω_r with ω. In this temporal eigenvalue problem, one seeks the four roots for the complex eigenvalue ω for specified values of N_ζ, N_{Re}, θ, and α_r.

In order to compare the predictions of the spatial and temporal formulations, it is necessary to convert the temporal amplification factor ω_i to an equivalent spatial amplification factor $-\alpha_i$. The most general transformation is that of Gaster (1965) given by

$$-\alpha_i = \omega_i/(\delta\omega_r/\delta\alpha_r) \qquad (9)$$

in which the partial derivative implies differentiation holding α_i constant. This transformation is restricted to weakly amplified disturbances for which $|\alpha_i| << \alpha_r$ and $|\omega_i| << \omega_r$. In the case of nondispersive disturbances for which $\delta(\omega_r/\alpha_r)/\delta\alpha_r = 0$, the above reduces to

$$-\alpha_i = \alpha_r\omega_i/\omega_r \qquad (10)$$

The latter transformation, suggested by Schubauer and Skramstad (1949), has been widely used to compare the predictions of the temporal growth theories with the measured properties of spatially growing disturbances, and is considerably easier to apply since it does not necessitate differentiating the temporal analogue of Equation (8).

Figure 1 shows the spatial amplification factor $-\alpha_i$ vs. the wave number α_r for the conditions of the highest Reynolds number data of Krantz and Goren (1971) shown by the circles. The solid, dotted, and dash-dotted lines correspond to the predictions of the spatial formulation given by Equation (8) and its temporal analogue transformed via Equations (9) and (10), respectively. The predictions of the spatial and temporal formulations are seen to be nonequivalent for some observable disturbances; moreover, the predictions of the spatial formulation are seen to be in closest agreement with the data. For unstable disturbances, the deviation between the predictions of the spatial formulation and those of the temporal formulation transformed via Equation (9) reaches a maximum of 2.3% at the most highly amplified wave number. For stable disturbances, this deviation progressively increases with increasing wave number; for example, at $\alpha_r = 1.0$, the deviation is 6.1%. The deviation betweeen the predictions of the spatial formulation and those of the temporal formulation transformed via Equation (10) is quite significant even for unstable waves, being 24% at the most highly amplified wave number. This more pronounced deviation arises because whereas Equation (9) only requires that the disturbances be weakly amplified, Equation (10) requires in addition that they be nondispersive. Disturbances in falling film flow are highly dispersive as is shown by Figure 2 which plots the phase velocity vs. wave number for the flow conditions of Figure 1. The phase velocity predicted by the temporal formulation, shown by the dash-dotted line, can be compared directly with that predicted by the spatial formulation, shown by the solid line, and again is seen to deviate slightly.

The predictions of the solutions developed here to the spatial and temporal formulations of this linear stability problem were compared over a wide range of values of the surface tension group N_ζ which characterizes the fluid. In all cases, when Equation (9) was employed, the deviation between the predictions of the two formulations never exceeded 5% for unstable disturbances. However, the deviation can, of course, be quite significant for stable disturbances when the decay rate becomes large.

Lin (1975*b*, 1976) has commented that the deviation between the spatial and temporal formulations is largest at wave numbers far removed from the neutrally stable wave number, where nonlinear effects might overshadow any such small deviations predicted by linear theory. Whereas in most practical applications this may be true, it does not obviate the need to answer the fundamental question as to whether there is any difference between the predictions of the spatial and temporal formulations of this linear stability problem. Note, however, that by carefully insulating a laboratory wetted wall column from room disturbances and by imposing disturbances of controlled amplitude and frequency, one can in fact obtain data for highly amplified small disturbances which grow (or decay) exponentially as would be predicted by linear theory. It was under such conditions that the data shown in Figures 1 and 2 were taken as has been discussed by Krantz (1975*a*). However, this suggests that the spatial growth of nonlinear waves may be an interesting area for further research, as very little appears to have been written on this subject.

It also has been noted by Lin (1975*b*) that the difference between the predictions of the spatial and temporal formulations [when transformed via Equation (9)] is smaller than the scatter in the data in Figures 1 and 2. This of course is true, and one cannot choose between the two formulations based on comparison with these data. Again, however, it must be stressed that the purpose of this note is to show that a systematic solution to the spatial and temporal formulations of this linear stability problem does not yield identically equivalent results. This can be demonstrated without recourse to any comparison with data. The data are presented only to support our claim that deviations between the two formulations exist for observable disturbances.

CONCLUSIONS

The systematic method of solution employed here indicates that the predictions of the spatial and temporal formulations of this linear stability problem are not identically equivalent for all observable disturbances, although the deviation is rather small at least for unstable disturbances. One might argue, as does Lin (1976), that if a higher-order version of Gaster's transformation had been developed by including more terms in the Taylor series expansion of the characteristic equation, then the deviation between the predictions of the two formulations would be negligible. This certainly may be true, depending of course on the radius of convergence of a proper Taylor series expansion of the characteristic equation about the selected point in $\alpha - \omega$ space. However, this has not been proven. Moreover, the primary purpose of this note is not to assess the range of validity of Gaster's transformation or any higher-order extension of it, but to demonstrate that the predictions of the spatial and temporal formulations are not equivalent. One formulation properly describes disturbances which grow or decay spatially, whereas the other describes disturbances which grow or decay temporally. Most experimental studies of the falling film problem have assumed that the disturbances grow temporally, when in fact they appear to grow spatially in most experiments. This note then stresses that the experimenter must determine whether the disturbances in his flow geometry exhibit spatial, temporal, or perhaps mixed mode growth or decay. Clearly, it is preferable to analyze the data emanating from stability experiments by using the predictions appropriate to the type of growth or decay encountered in the particular flow geometry. This obviates the need to employ Equation (9) which is neither trivial to apply nor exact for all but very weakly amplified disturbances.

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NOTATION

H = basic film thickness
N_{Re} = Reynolds number $= \overline{U}H/\nu$
N_{ζ} = surface tension group $= (\sigma/\rho)(3/g\nu^4)^{1/3}$
t = dimensionless time $= t^*\overline{U}/H$
U = dimensionless velocity of basic flow $= (3/2)(1 - y^2)$
$\overline{U}$ = average velocity of basic flow
x = dimensionless streamwise coordinate $= x^*/H$
y = dimensionless cross-stream coordinate $= y^*/H$

Greek Letters

α = dimensionless complex wave number $= \alpha^*H$
α_i = dimensionless spatial amplification factor
α_r = dimensionless real wave number
θ = angle of inclination of the plane to the horizontal
ν = kinematic viscosity
ρ = density
σ = surface tension
ϕ = dimensionless amplitude of the stream function $= \phi^*/\overline{U}H$
ψ = dimensionless stream function $= \psi^*/\overline{U}H$
ω = dimensionless complex angular frequency $= \omega^*H/\overline{U}$
ω_i = dimensionless temporal amplification factor
ω_r = dimensionless real angular frequency

Superscripts

$'$ = derivative with respect to y
$*$ = dimensional quantity

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On the Applicability of the Linear Stability Theory

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Shuler and Krantz claim in the previous article that the predictions of the spatial and temporal formulations are nonequivalent and that the predictions of the spatial formulation are in closer agreement with the observed data. The present author cannot but disagree completely for the following reasons. First of all, it should be pointed out that the linear stability theory, be it based on temporal or spatial formulation, is valid only if the neglected nonlinearity remains small. Thus the linear theory can be used to predict only the onset of instability and its subsequent exponential growth of the disturbances as long as the neglected nonlinear terms remain small. Note that at the onset of instability, $\omega_i = o$ for the temporal case and $\alpha_i = o$ for the spatial case. Thus the linear theory remains valid over a time T or a distance X such that $[\exp(\omega_i T) - \exp(o)] << 1$ or $[\exp(-\alpha_i X) - \exp(o)] << 1$. Therefore, for $T = 0(1)$, $X = 0(1)$, the linear theory is valid only if $|\omega_i| << 1$ or $|\alpha_i| << 1$, regardless of the values of α_r. Note that when $|\omega_i| << 1$ or $|\alpha_i| << 1$, the amplification rate of the disturbance, be it short shear waves or long gravity waves, remains small. Under this condition, Gaster (1962) proved rigorously that the temporal growth rate and the spatial growth rate are related by

$$\omega_i = -\alpha_i(\partial\omega_r/\partial\alpha_r) - \tfrac{1}{2}\alpha_i^2(\partial^2\omega_r/\partial\alpha_r\partial\alpha_i)$$

The last term in the above equation is omitted in the most general transformation of Gaster given by (9) of Shuler and Krantz. The above equation states that the two formulations give identical neutral curves and equivalent amplification rates. The above equivalence was rigorously